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Supersymmetry and Lorentzian holonomy in various dimensions

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Abstract

We present a systematic method for constructing manifolds with Lorentzian holonomy group that are non-static supersymmetric vacua admitting covariantly constant light-like spinors. It is based on the metric of their Riemannian counterparts and the realization that, when certain conditions are satisfied, it is possible to promote constant moduli parameters into arbitrary functions of the light-cone time. Besides the general formalism, we present in detail several examples in various dimensions.

1 Introduction

Ricci-flat manifolds admitting covariantly constant spinors are purely gravitational supersymmetric solutions of M-theory which imply the existence of parallel spinors and constrain the holonomy group of the manifold. In particular, the maximum number of preserved supersymmetries is given by the number of singlets arising from the decomposition of the spinor representation of $Spin(10, 1)$ under the holonomy group of the manifold. When the Killing vector constructed from the Killing spinor is time-like, the corresponding spacetimes are static vacua, and their Riemannian holonomy groups have been completely classified [1]. Non-static vacua correspond to spacetimes with a covariantly constant light-like vector, and although the holonomy groups of Lorentzian manifolds have not been classified, it is already known which subgroups of $Spin(10, 1)$ leave a null spinor invariant [2]. Static spacetimes in Euclidean signature satisfying the supergravity equations of motion are automatically Ricci-flat. However, in Lorentzian signature supersymmetric vacua are only required to be Ricci-null [3], and Ricci-flatness therefore needs to be imposed as an additional condition.

Examples of supersymmetric Ricci-null manifolds with Lorentzian holonomy have been discussed in [2, 3], extending a solution already considered in [4]. However, in spite of their great physical interest there has been no systematic approach to the construction of metrics with Lorentzian holonomy. In this note we will try to fill this gap constructing manifolds with reduced Lorentzian holonomy group in diverse dimensions using the explicitly known form of the metric for some of their Riemannian counterparts. A possible way to include time-dependence is by allowing the moduli parameters of a Riemannian metric to be arbitrary functions of the light-cone time. Although this might seem plausible, it is far from trivial to preserve at the same time a fraction of maximal supersymmetry and the vacuum character of the original solution. The present paper is organized as follows: In section 2 we will develop the general formalism using the supergravity and Killing spinor equations to obtain D-dimensional vacuum supersymmetric solutions with Lorentzian holonomy group of the semi-direct product type $\mathbb{G} \ltimes \mathbb{R}^{D-2}$. In section 3 we will present explicitly several manifolds with Lorentzian holonomy in six, eight and nine dimensions, with $\mathbb{G} = SU(2)$, $SU(3)$ and G_2, respectively. To do so it will be enough to utilize a simplified version of our method. In section 4 we conclude with some remarks and comments on future work. In an appendix a general six-dimensional solution based on the four-dimensional self-dual

Gibbons–Hawking multi-center metrics is constructed making use of the method in its full generality.

2 Ricci-null manifolds

In this paper we will construct and analyze in detail D-dimensional metrics admitting parallel null spinors of the general form

$$d\hat{s}_D^2 = 2du\,dv + 2V_i(u,x)du\,dx^i + F(u,x)du^2 + g_{ij}(u,x)dx^i dx^j \ , \ \ i,j = 1,2,\ldots,D-2 \ . \tag{2.1}$$

When the light-cone time u is treated as just a parameter, the transverse space metrics $g_{ij}(u,x)$ can be taken to be a family of metrics with holonomy contained in some group $\mathbb{G} \in SO(D-2)$. We will be particularly interested in cases where $D-2 = 4, 6$ and 7, with the transverse metrics respectively having an $SU(2)$, $SU(3)$ and G_2 holonomy group. The functions V_i and F will be determined by requiring that the full metric (2.1) is Ricci-flat and also preserves a fraction of the sypersymmetry preserved by the transverse metric g_{ij}. Then (2.1) will provide non-static M-theory vacua of the form $ds_{11}^2 = (dx^a)^2 + d\hat{s}_D^2$, where $a = 1, \ldots, 11-D$. We will use the frame basis

$$\hat{e}^+ = du \ , \qquad \hat{e}^- = dv + V_i dx^i + \frac{F}{2}du \ , \qquad \hat{e}^a = e_i^a(u,x^i)dx^i \ , \tag{2.2}$$

where e_i^a is the corresponding basis for the transverse metric. Then, the only non-vanishing components of the spin connection are

$$\begin{aligned}
\hat{\omega}^{ab} &= \omega^{ab} - \frac{1}{2}\Big(\dot{e}^{[a}e^{b]i} + e^{ai}e^{bj}\partial_{[i}V_{j]}\Big)du \ , \\
\hat{\omega}^{a-} &= \frac{1}{2}\dot{e}_i^a dx^i + \frac{1}{2}\dot{e}_i^b e^{ai} e_b - \frac{1}{2}\left(\partial_i F - 2\dot{V}_i\right)e^{ai}du - \frac{1}{2}e^{ai}\partial_{[i}V_{j]}dx^j \ ,
\end{aligned} \tag{2.3}$$

where the dot stands for partial derivatives with respect to u, and ω^{ab} is the spin connection for $g_{ij}(u,x)$. The Killing spinor equation for purely gravitational backgrounds is

$$\partial_\mu \epsilon + \frac{1}{4}\hat{\omega}_\mu^{AB}\Gamma_{AB}\epsilon = 0 \ , \qquad A = (+,-,a) \ . \tag{2.4}$$

We will now assume that the transverse background with metric $g_{ij}(u,x)$, where u is treated as a parameter, satisfies the Killing spinor equation

$$\partial_i \epsilon + \frac{1}{4}\omega_i^{ab}\Gamma_{ab}\epsilon = 0 \ , \tag{2.5}$$

which has non-trivial solutions provided a closed set of projections, involving products of Gamma-matrices, are imposed on the Killing spinor. Hence, in this way, a certain fraction of supersymmetry will be preserved according to the choice of transverse metrics $g_{ij}(u,x)$. In addition, with the help of these projections the Killing spinor satisfying (2.5) will be left invariant under the action of the holonomy group of this manifold $\mathbb{G} \in SO(D-2)$.

In order to solve (2.4) we further impose the additional projection

$$\Gamma^{+}\epsilon = 0 \ , \tag{2.6}$$

and simultaneously require that the deviation of $\hat{\omega}^{ab}$ from the transverse space connection ω^{ab} is proportional to a matrix Λ^{ab}, such that the net result on the Killing spinor is zero. Namely, we define

$$\Lambda^{ab} = \dot{e}_i^{[a} e^{b]i} + e^{ai} e^{bj} \partial_{[i} V_{j]} \tag{2.7}$$

and demand that

$$\Lambda_{ab}\Gamma^{ab}\epsilon = 0 \ . \tag{2.8}$$

In order for the last condition to be satisfied the left hand side should be a linear combination of the bilinears in Gamma-matrices that act as projection operators on the spinor, and guarantee the existence of the Killing spinor corresponding to the transverse space metric g_{ij}. The equivalent relation to (2.7),

$$\Lambda_{ab} e_i^a e_j^b = \partial_{[i} V_{j]} - \dot{e}^a_{[i} e^a_{j]} \ , \tag{2.9}$$

will also prove useful. Then from the Killing spinor equation (2.4) and the spin connection (2.3) we can see that the manifold (2.1) will preserve half of the supersymmetries preserved by compactifications along $g_{ij}(u,x)$. The Killing spinor will therefore be the same as that solving equation (2.5), and in particular $\partial_u \epsilon = 0$. We must emphasize that it is highly non-trivial that we may introduce a u-dependence in the transverse metric basis e_i^a such that with an appropriate choice of a set of functions V_i a matrix Λ^{ab} exists, such that the condition (2.8) is satisfied.

In general, from the integrability condition of the Killing spinor equation (2.4) it follows that

$$\hat{R}_{AB}\hat{R}_{AC}\eta^{BC}\epsilon = 0 \ , \quad \forall\, A \ , \tag{2.10}$$

which, is the condition for a Ricci-null manifold. Unlike the Euclidean case, in the Lorentzian one this does not automatically imply Ricci-flatness, i.e. that $\hat{R}_{AB} = 0, \forall\, A, B$.

Indeed, specializing in (2.10) to the cases $A = a$ and $A = +$ we find that $\hat{R}_{ab} = \hat{R}_{a+} = 0$, but there is no information on the component $\hat{R}_{++}$.

Returning to our metric ansatz (2.1), the vanishing of $\hat{R}_{ab}$ is apparent from the fact that $\hat{\omega}^{a+} = 0$ and that $R_{ab} = 0$ by the assumption (2.5) for a supersymmetric transverse metric g_{ij}. However, the vanishing of $\hat{R}_{a+}$ occurs in a non-trivial way as it involves properties of the matrix Λ^{ab} and, in particular, (2.8). Hence, we find it worth to present some necessary steps. First, using the spin connection we obtain the following explicit expression

$$\hat{R}_{a+} = \dot{\omega}_i^{ab} e_b^i + \frac{1}{2} e_b^i \, D_i \Lambda_a{}^b \ , \tag{2.11}$$

where we remind the reader that the covariant derivative $D_i \Lambda_a{}^b$ contains the spin connection ω^{ab}. Then by taking the u-derivative of the Killing spinor equation (2.5) we have

$$\dot{\omega}_i^{ab} \Gamma_{ab} \epsilon = 0 \ . \tag{2.12}$$

Let us now contract this relation with $e^{ic}\Gamma_c$, and use $\Gamma_c \Gamma_{ab} = \Gamma_{cab} - \Gamma_{[a}\delta_{b]c}$. This gives

$$\dot{\omega}_i^{cb} e_c^i \Gamma_b \epsilon = -\frac{1}{2} \dot{\omega}_i^{ab} e^{ic} \Gamma_{abc} \epsilon = \frac{1}{4} e^{ia} D_i \Lambda^{bc} \Gamma_{abc} \epsilon \ , \tag{2.13}$$

where the last step follows after several algebraic manipulations using the explicit expression of the spin-connection in terms of the frame basis components e_i^a together with the condition (2.9). Then we have

$$\begin{aligned} \hat{R}_{a+} \Gamma^a \epsilon &= -\frac{1}{4} e^{ia} D_i \Lambda^{bc} \Gamma_{abc} \epsilon + \frac{1}{2} e_b^i D_i \Lambda^{ab} \Gamma_a \epsilon \\ &= -\frac{1}{4} e^{ia} \Gamma_a (D_i \Lambda^{bc} \Gamma_{bc} \epsilon) = 0 \ . \end{aligned} \tag{2.14}$$

The last equality follows from the Killing spinor equation (2.5) in combination with condition (2.8). From the above it easily follows that $\hat{R}_{a+} = 0$.

We have seen that Ricci-flatness demands setting the $\hat{R}_{++}$ component to zero. This will provide a second order differential equation for F, which is found to be

$$D_g^2 F = 2 g^{ij} D_i \dot{V}_j - 2 e_a^i \ddot{e}_i^a - 2 \dot{e}_i^a e^{bi} \Lambda^{ab} + \frac{1}{2} \Lambda^{ab} \Lambda_{ab} \ , \tag{2.15}$$

where we have made use of (2.7), and D_g^2 is the Laplacian corresponding to the metric $g_{ij}(u, x)$, in which u is treated as a parameter.

The holonomy group of the metric (2.1) is contained in $\mathbb{G} \ltimes \mathbb{R}^{D-2}$, which will be generated by bilinears in the Gamma-matrices of the form $J^{ab} = M^{ab}{}_{cd} \Gamma^{cd}$ and $J^a = \Gamma^{a+}$.

The constants $M^{ab}{}_{cd}$ help to project the generators Γ^{ab} of $SO(D-2)$ into the independent generators J^{ab} of the Lie algebra of $\mathbb{G}$. The J^a's commute among themselves due to the nilpotency of Γ^+ and form a closed set under the commutator with J^{ab}, with $M^{ab}{}_{cd}$ providing the necessary structure constants.

The most general way to introduce a time-dependence is to promote the moduli parameters of the transverse space metric $g_{ij}(u,x)$, arising as integration constants in solving the Killing spinor equations (2.5), into arbitrary functions of the variable u. The appearance of the moduli parameters in a solution is ultimately related to its regularity and to the amount of symmetry that it preserves. Our method provides a dynamical way to change the symmetry and singularity structure of a given solution that is controllable and also consistent with supersymmetry.

There has been in the past works where some solutions of low energy effective string theory were reinterpreted as if they were originating from solutions in one less dimension, in which some moduli parameters were made to depend on the extra dimension (see, for instance, [5]-[8]). Here we have required that the original solution is supersymmetric and moreover that a fraction of supersymmetry is preserved when the moduli parameters depend on the extra coordinates. It is this important extra ingredient that makes it necessary, as the minimal choice, that the moduli parameters are functions of the *light-cone* time, instead of just the additional ordinary time-like or space-like coordinate.

When it comes to explicit examples, in this paper we will be mainly interested in cases where a u-dependence can be introduced in the most minimal way possible. In particular, in cases where the choice

$$V_i = 0 \ , \qquad \Lambda^{ab} = 0 \ , \tag{2.16}$$

can be made, and only the function F remains to be determined. Then conditions (2.7)-(2.9) simplify to

$$\dot{e}_i^{[a} e^{b]i} = 0 \quad \Longleftrightarrow \quad \dot{e}^a_{[i} e^a_{j]} = 0 \ , \tag{2.17}$$

while equation (2.15) reduces to

$$D_g^2 F = -2 e^i_a \ddot{e}^a_i \ . \tag{2.18}$$

Even within this subclass we may make yet another simplification, which nevertheless leads to non-trivial results. Namely, by introducing the u-dependence only in the overall length scale, which then becomes dynamical. In particular, if $e^a_i(u,x) = \Omega(u)\tilde{e}^a_i(x)$, the condition

(2.17) is trivially satisfied and (2.18) becomes

$$D_{\tilde{g}}^2 F = -2(D-2)\Omega\ddot{\Omega} \ . \tag{2.19}$$

In the appendix we will construct an example where the minimal choice (2.16) is not valid and instead the method we have presented has to be used in its full generality.

3 Supersymmetric waves with Lorentzian holonomy

For definiteness, we will now present some particular examples corresponding to manifolds with Lorentzian holonomy group in various dimensions. We will first construct six-dimensional metrics with $SU(2) \ltimes \mathbb{R}^4$ holonomy, by taking the transverse metric $g_{ij}(u,x)$ to have $SU(2)$ holonomy. Subsequently, we will construct examples in eight and nine dimensions having $SU(3) \ltimes \mathbb{R}^6$ and $G_2 \ltimes \mathbb{R}^7$ holonomy, respectively. Ten-dimensional manifolds with $SU(4) \ltimes \mathbb{R}^8$ and $Spin(7) \ltimes \mathbb{R}^8$ holonomy can be constructed along the same lines, but we will not describe them here.

3.1 Six dimensions

Let us first consider a six-dimensional case preserving 1/4 of the maximal number of supercharges. Now $g_{ij}(u,x)$ must be a four-dimensional hyper-Kähler manifold, with $SU(2)$ holonomy group. In the appendix we will consider a general multi-center solution, but in this section we will simply take the four-dimensional seed solution with, generically, $SU(2)$ isometry,

$$ds_4^2 = f^2(r)dr^2 + a_1^2(r)\sigma_1^2 + a_2^2(r)\sigma_2^2 + a_3^2(r)\sigma_3^2 \ , \tag{3.1}$$

where it is particularly convenient to choose the function $f = a_1 a_2 a_3$, and where σ_i are left-invariant Maurer-Cartan $SU(2)$ 1-forms taken to satisfy

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k \ . \tag{3.2}$$

Using the natural frame

$$e^i = a_i\sigma_i \ , \qquad i = 1,2,3 \ , \qquad e^4 = f dr \tag{3.3}$$

and imposing on the spinor the projection

$$\Gamma_4\epsilon = -\Gamma_{123}\epsilon \ , \tag{3.4}$$

one finds that the metric ansatz (3.1) breaks 1/2 of the maximal supersymmetry and the functions a_i satisfy a non-linear system of equations which was first derived by imposing the self-duality condition for the Riemann curvature for the class of metrics (3.1) in [9]. This is the (Euclidean) Euler system if the Killing spinor is a constant, and the Halphen system when the Killing spinor depends on the $SU(2)$ group manifold variables.

Introducing the u-dependence we will obtain a six-dimensional solution with metric

$$ds_6^2 = 2\,du\,dv + F(u,r)du^2 + f^2(u,r)dr^2 + a_1^2(u,r)\sigma_1^2 + a_2^2(u,r)\sigma_2^2 + a_3^2(u,r)\sigma_3^2 \ , \qquad (3.5)$$

where we have already restricted in our ansatz to a function F that is independent of the $SU(2)$ group manifold variables. Then (2.18) becomes

$$F'' = -4f^2\Big(\frac{\ddot a_1}{a_1} + \frac{\ddot a_2}{a_2} + \frac{\ddot a_3}{a_3} + \frac{\dot a_1 \dot a_2}{a_1 a_2} + \frac{\dot a_1 \dot a_3}{a_1 a_3} + \frac{\dot a_2 \dot a_3}{a_2 a_3}\Big) \ , \qquad (3.6)$$

where the prime denotes the partial derivatives with respect to r. In the simpler case of an overall u-dependent length scale, $a_i(u,r) = \Omega(u)\tilde a_i(r)$, the condition (2.19) gives

$$F'' = -8\Omega\ddot\Omega \tilde f^2 \ , \qquad (3.7)$$

where now $\tilde f = \tilde f(r)$.

The holonomy group in general is computed by looking at the independent generators among the combinations $\hat R_{AB}{}^{CD}\Gamma_{CD}$ and making sure that they form a closed algebra. For the six-dimensional metric (3.5) the holonomy group is found to be $SU(2) \ltimes \mathbb{R}^4$ and the explicit form of the independent generators is

$$J^i = \Gamma^{4i} - \frac{1}{2}\epsilon_{ijk}\Gamma^{jk} \ , \qquad i = 1,2,3 \ , \qquad \Gamma^{a+} \ , \qquad a = (i,4) \ . \qquad (3.8)$$

Note that (3.4), together with the universal projection (2.6), restricts the Killing spinor to be an invariant of the holonomy group.

Among the most important examples that can be constructed in this case are those involving the Eguchi–Hanson and the Taub–NUT metrics.

3.1.1 Eguchi–Hanson

In this case we have [12]

$$a_1^2(r) = a_2^2(r) = m^2\coth(m^2 r) \ , \qquad a_3^2(r) = \frac{2m^2}{\sinh(2m^2 r)} \ , \qquad r \geq 0 \ , \qquad (3.9)$$

where m is the moduli parameter. At $r \to \infty$ there is a removable singularity of the *bolt* type, and the asymptotic region is located at $r \to 0$. The lift to eleven dimensions of the static limit of this manifold is $ds_{11}^2 = dx_{1,6}^2 + ds_{EH}^2$, and corresponds to a solution describing a set of flat D6-branes. When the constant moduli parameter m is replaced by a function $m(u)$ we have from (3.6) that

$$\begin{aligned} F(u,r) &= 2\dot{m}^2 \left[\coth(m^2 r) + \frac{m^2 r}{\sinh^2(m^2 r)} - 2m^4 r^2 \frac{\coth(m^2 r)}{\sinh^2(m^2 r)}\right] \\ &\quad + 2m\ddot{m} \left[\frac{m^2 r}{\sinh^2(m^2 r)} - \coth(m^2 r)\right] , \qquad m = m(u) , \end{aligned} \tag{3.10}$$

where a function of u coming from the integration has been set to zero in order to avoid a linearly increasing behaviour of $F(u,r)$ as $r \to \infty$. The six-dimensional metric is regular since for large r, near the *bolt*, the function $F \simeq 2\dot{m}^2 - 2m\ddot{m}$ and therefore it can be absorbed by a shift of the coordinate v. Hence, the six-dimensional space near $r \to \infty$ looks like $S^2 \times \mathbb{R}^2$ times the two-dimensional light-cone. In the asymptotic region at $r \to 0$ the function $F \simeq -\frac{4}{3} m^3 \ddot{m} r \to 0$, and we get the four-dimensional Euclidean space, as a cone over $S^3/\mathbb{Z}_2$, times the two-dimensional light-cone. Hence we interpolate between these two spaces of different topology as the variable r exhausts its full range from $r = 0$ to $r \to \infty$. This interpolation can also happen in a dynamical way if we choose for the function $m(u)$ the following specific behaviour in the remote past and future for the light-cone time u,

$$\lim_{u \to -\infty} m = 0 , \qquad \lim_{u \to \infty} m \to \infty . \tag{3.11}$$

At the end points the symmetry group for the metric (3.5) gets enhanced, as a result of its evolution in time (light-cone). Another possibility is to have functions behaving as $\lim_{u \to \pm\infty} m = m_{\pm}$. In that case we obtain in the remote past and remote future the direct product of the four-dimensional Eguchi–Hanson metric with the two-dimensional light-cone.[1]

For the simpler situation where the u-dependence is via an overall function in the metric, $\Omega = \Omega(u)$, we easily conclude from (3.7) that

$$F(u,r) = -4\Omega\ddot{\Omega}\, m^2 \coth(m^2 r) , \qquad m = \text{const.} , \tag{3.12}$$

where again a function arising from the integration has been chosen such that there is no growing behaviour in the limit $r \to \infty$. As before near the *bolt* the solution looks like

[1]Similar remarks can be made for the other explicit examples we present in this paper.

$S^2 \times \mathbb{R}^2$ times the two-dimensional light-cone. In the asymptotic region $r \to 0$ the function F blows up as $1/r$.

3.1.2 Taub–NUT

The Taub–NUT metric corresponds to [13][2]

$$a_1^2(r) = a_2^2(r) = \frac{1+4m^2 r}{4m^2 r^2} \ , \qquad a_3^2(r) = \frac{4m^2}{1+4m^2 r} \ , \qquad r \geq 0 \ , \tag{3.13}$$

where m is the moduli parameter. The removable *nut* singularity is located at $r \to \infty$, and the asymptotic region at $r \to 0$. For the case where the constant moduli parameter m is replaced by a function $m(u)$ we conclude from (3.6) that

$$F(u,r) = \frac{m\ddot{m} - 3\dot{m}^2}{6m^4 r^2} \ , \qquad m = m(u) \ . \tag{3.14}$$

When the u-dependence is via an overall function in the metric, from (3.7) we obtain

$$F(u,r) = -\Omega\ddot{\Omega}\left(\frac{4}{r} + \frac{1}{3m^2 r^2}\right) \ , \qquad m = \text{constant} \ , \tag{3.15}$$

where again in both cases a function of u arising from the integration has been chosen such that there is no growing behaviour as $r \to \infty$. In both cases the function F vanishes near the *nut* singularity as $r \to \infty$, and blows up asymptotically when $r \to 0$.

3.2 Eight dimensions

A family of metrics with $SU(3) \ltimes \mathbb{R}^6$ holonomy, preserving 1/8 supercharges, arises when $g_{ij}(u,x)$ is a Calabi-Yau threefold with $SU(3)$ holonomy group. The six-dimensional seed solution that we will employ has the form

$$ds_6^2 = f^2(r)dr^2 + a_1^2(r)(\sigma_1^2 + \sigma_2^2) + a_2^2(r)(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + b^2(r)(\sigma_3 + \hat{\sigma}_3)^2 \ , \tag{3.16}$$

where σ_i and $\hat{\sigma}_i$, with $i = 1,2$, are now triplets of Maurer–Cartan 1-forms on a two-sphere, normalized so that they again obey (3.2). Using the natural frame

$$e^i = a_1 \sigma_i \ , \quad e^{\hat{i}} = a_2 \hat{\sigma}_i \ , \quad i = 1,2 \ , \qquad e^3 = b(\sigma_3 + \hat{\sigma}_3) \ , \qquad e^4 = f dr \ , \tag{3.17}$$

[2]The holonomy of the purely geometrical solution constructed by combining a wave with the Taub–NUT space [10, 11] has been worked out in [14].

and imposing the two projections (see, for instance, [15])

$$\Gamma_{12}\epsilon = \hat{\Gamma}_{12}\epsilon = -\Gamma_{34}\epsilon\ , \tag{3.18}$$

one finds a set of differential equations for the coefficients, and the metric (3.16) breaks 1/4 of the maximal supersymmetry. The explicit solution is the so-called resolved conifold metric [16] with

$$\begin{aligned} a_1^2(r) &= \frac{r^2+6a^2}{6}\ , \qquad a_2^2(r) = \frac{r^2}{6}\ , \\ f^2(r) &= \frac{r^2+6a^2}{r^2+9a^2}\ , \qquad b^2(r) = \frac{r^2}{9f^2(r)}\ . \end{aligned} \tag{3.19}$$

The limit $a \to 0$ corresponds to the singular conifold. The u-dependent eight-dimensional metric will then be

$$\begin{aligned} ds_8^2 &= 2\,du\,dv + F(u,r)du^2 + f^2(u,r)dr^2 + a_1^2(u,r)(\sigma_1^2+\sigma_2^2) \\ &\quad + a_2^2(u,r)(\hat{\sigma}_1^2+\hat{\sigma}_2^2) + b^2(u,r)(\sigma_3+\hat{\sigma}_3)^2\ , \end{aligned} \tag{3.20}$$

where again the function F is taken to be independent of the $SU(2)$ group manifold variables. From the condition (2.18) we now get the differential equation

$$\left(f^{-1}a_1^2a_2^2bF'\right)' = -2fa_1^2a_2^2b\left(\frac{\ddot{f}}{f} + 2\frac{\ddot{a}_1}{a_1} + 2\frac{\ddot{a}_2}{a_2} + \frac{\ddot{b}}{b}\right)\ . \tag{3.21}$$

When the resolved conifold moduli is replaced by a u-dependent function, $a = a(u)$, we obtain from (3.21) and (3.19) the explicit solution

$$F(u,r) = -27\frac{a^2\dot{a}^2}{r^2+9a^2} - 3(\dot{a}^2 + a\ddot{a})\ln(r^2+9a^2)\ , \tag{3.22}$$

where a function of u arising from the integration has been chosen so that there is no $1/r^2$ term.

When the u-dependence comes through a conformal factor, the differential equation for F (2.19) simplifies to

$$\left(f^{-1}a_1^2a_2^2bF'\right)' = -12\Omega\ddot{\Omega}fa_1^2a_2^2b\ . \tag{3.23}$$

Using (3.19) we can solve for F to obtain

$$F(u,r) = -\Omega\ddot{\Omega}r^2 - \frac{c(u)}{18a^2r^2} + \frac{c(u)}{162a^4}\ln\left(1+\frac{9a^2}{r^2}\right)\ , \tag{3.24}$$

where $c(u)$ is a function arising from the integration.

The holonomy group of the metric (3.20) is $SU(3) \ltimes \mathbb{R}^6$, and the explicit form of the independent generators is found to be

$$
\begin{aligned}
J_1 &= \Gamma_{\hat{1}2} + \Gamma_{1\hat{2}}, \qquad J_2 = \Gamma_{2\hat{2}} + \Gamma_{1\hat{1}}, \qquad J_3 = \Gamma_{\hat{1}\hat{2}} - \Gamma_{12}, \\
J_4 &= \Gamma_{42} + \Gamma_{13}, \qquad J_5 = \Gamma_{23} - \Gamma_{41}, \qquad J_6 = \Gamma_{3\hat{1}} - \Gamma_{4\hat{2}}, \\
J_7 &= \Gamma_{4\hat{1}} - \Gamma_{\hat{2}3}, \qquad J_8 = 3^{-1/2}(2\Gamma_{43} - \Gamma_{12} - \Gamma_{\hat{1}\hat{2}}) \ ,
\end{aligned}
\tag{3.25}
$$

together with Γ^{a+}. We can easily verify that the projections (3.18) and (2.6) guarantee that the Killing spinor is an invariant of the holonomy group.

3.3 Nine dimensions

Let us now take $g_{ij}(u,x)$ to be a seven-manifold of G_2 holonomy, with an $SU(2) \times SU(2)$ isometry. The seed solution will be the seven-dimensional metric ansatz

$$
ds_7^2 = dr^2 + \sum_{i=1}^{3} a_i^2(r)\sigma_i^2 + \sum_{i=1}^{3} b_i^2(r)\Big(\Sigma_i + c_i(r)\sigma_i\Big)^2 \ , \tag{3.26}
$$

where the σ_i's and Σ_i's are two different sets of $SU(2)$ Maurer–Cartan 1-forms obeying the normalization condition (3.2). Using the natural frame

$$
e^i = a_i\sigma_i \ , \quad e^{\hat{i}} = b_i(\Sigma_i + c_i\sigma_i) \ , \quad e^7 = dr \ , \quad i = 1,2,3 \ , \quad \hat{i} = i + 3 \ , \tag{3.27}
$$

and imposing the three independent projections (see, for instance, [15]-[19])

$$
\Gamma_{ij}\epsilon = -\Gamma_{\hat{i}\hat{j}}\epsilon \ , \qquad \Gamma_7\epsilon = -\Gamma_{\hat{1}\hat{2}\hat{3}}\epsilon \ , \tag{3.28}
$$

one finds a first order system of coupled differential equations for the coefficients, and the metric (3.16) breaks 1/8 of the maximal supersymmetry. The functions c_i are not independent: they are given as rational functions of the a_i's and b_i's (for details see equations (30) and (31) of [18]). In particular, a useful property of the c_i's is that they do not scale when all functions a_i and b_i are scaled by the same factor. Allowing now all functions to depend on the light-cone time u we obtain an M-theory vacuum preserving 1/16 supercharges, with non-trivial nine-dimensional metric with $G_2 \ltimes \mathbb{R}^7$ holonomy group given by

$$
ds_9^2 = 2\,du\,dv + F(u,r)du^2 + dr^2 + \sum_{i=1}^{3} a_i^2(u,r)\sigma_i^2 + \sum_{i=1}^{3} b_i^2(u,r)\Big(\Sigma_i + c_i(u,r)\sigma_i\Big)^2 \ . \tag{3.29}
$$

However, applying condition (2.17) to this metric we get the constraint $\sum_{i=1}^{3} \frac{a_i \dot{c}_i}{b_i} = 0$, which is satisfied when $\dot{c}_i = 0$, for $i = 1, 2, 3$. In particular we can make the functions c_i independent of u by allowing only u-dependence as an overall factor in the metric, because, as mentioned before, they do not scale when we scale all the coefficients a_i and b_i by the same factor. An exception to this is the round case where all a_i are equal to each other, and similarly for the functions b_i, since then $c_i = -1/2$, for $i = 1, 2, 3$. This is the manifold we will consider in detail.[3]

In the round case the isometry group of g_{ij} develops an additional $SU(2)$ factor, and the form of the coefficients in (3.29) is explicitely known [20]. In this case, promoting the moduli parameter into a function of u, we have

$$ds_9^2 = 2\,du\,dv + F du^2 + \Big(1 - \frac{\rho_0^3(u)}{\rho^3}\Big)^{-1} d\rho^2 + \frac{\rho^2}{12}\sum_{i=1}^{3} \sigma_i^2 + \frac{\rho^2}{9}\Big(1 - \frac{\rho_0^3(u)}{\rho^3}\Big)\sum_{i=1}^{3}\Big(\Sigma_i - \frac{1}{2}\sigma_i\Big)^2 \,, \tag{3.30}$$

and from equation (2.18) we can solve for F. We find that

$$F' = \frac{6\rho^4}{(\rho^3 - \rho_0^3(u))^2}\,\partial_u\Big(\rho_0^2\,\dot{\rho}_0\Big) + \frac{c(u)\rho^3}{(\rho^3 - \rho_0^3(u))^2} \,, \tag{3.31}$$

where $c(u)$ arises from the integration. The form of $F(u,\rho)$ can be readily obtained from (3.31), but it is a complicated expression and we will not present it here.

If we consider instead the case where all u-dependence in the metric is through a conformal factor we find that

$$F(u,r) = -\frac{7}{12}\Omega\ddot{\Omega}\left[3\rho\left(\rho + \frac{2\rho_0^3}{\rho^2 + \rho_0\rho + \rho_0^2}\right) - 4\sqrt{3}\rho_0^2\tan^{-1}\left(\frac{2\rho + \rho_0}{\sqrt{3}\rho_0}\right)\right] \,, \tag{3.32}$$

where an integration constant has been chosen such that F remains finite as $\rho \to \rho_0$.

We finally note that the generators of the holonomy group $G_2 \ltimes \mathbb{R}^7$, labeled according to (3.27), are given by

$$\begin{aligned}
&\Gamma_{ij} + \Gamma_{\hat{i}\hat{j}} \,, \qquad \Gamma_{i\hat{j}} - \Gamma_{\hat{i}j} \,, \\
&2\Gamma_{7i} + \epsilon_{ijk}\Gamma_{j\hat{k}} \,, \qquad 2\Gamma_{7\hat{i}} + \frac{1}{2}\epsilon_{ijk}(\Gamma_{jk} - \Gamma_{\hat{j}\hat{k}}) \,, \\
&\Gamma_{1\hat{1}} - \Gamma_{2\hat{2}} \,, \qquad \Gamma_{2\hat{2}} - \Gamma_{3\hat{3}} \,.
\end{aligned} \tag{3.33}$$

[3]It would be interesting to investigate if the general metric (3.26) is consistent with supersymmetry and remains a vacuum solution by employing the general construction via (2.1) and (2.7), that has non-vanishing V_i and Λ^{ab}. On the contrary, the metrics in [21] with an $SU(2) \times SU(2) \times \mathbb{Z}_2$ isometry, although not in the class of metrics (3.26), can be employed to construct, within the minimal choice (2.16), a nine-dimensional manifold with Lorentzian holonomy.

together with Γ^{a+}. The generators of the G_2 algebra as bilinears of the Gamma-matrices are in agreement with a similar expression in [22]. In addition, up to overall rescalings, these can be assembled in

$$J^{ab} = \Gamma^{ab} + \frac{1}{4}\psi_{abcd}\Gamma^{cd} \ , \qquad J^{a} = \Gamma^{a+} \ , \tag{3.34}$$

where ψ_{abcd} is the 4-index G_2-invariant tensor constructed as the dual of the octonionic structure constants [23].[4] Let us also note that the projections (3.28), together with (2.6), imply that the Killing spinor is indeed invariant under the action of the holonomy group.

4 Conclusions

In this paper we have presented a systematic approach to Ricci-null manifolds with reduced Lorentzian holonomy group. The manifolds that we have constructed are supersymmetric waves in six, eight and nine dimensions, with a transversal Riemannian metric and explicit dependence on the light-cone time. An interesting feature of our metrics, when the temporal dependence arises from variable moduli parameters, is that as the light-cone time evolves their isometry group can be modified.

We have only covered purely gravitational solutions to the equations of motion of eleven-dimensional supergravity. A more precise understanding of M-theory requires a complete picture of all possible supersymmetric vacua, and must include configurations with non-vanishing four-form flux. In the presence of background flux an important differences arises: Killing spinors are no longer covariantly constant with respect to the Levi–Civita connection, as in the purely gravitational case, but with respect to a connection on the spin bundle, which gives rise to a generalized holonomy group [25, 26]. In fact, supersymmetric solutions with non-vanishing four-form flux are not Ricci-flat, and have already been classified (see, for instance, [27, 28, 14]). An interesting extension of the present work is the deformation of our solutions to include a non-vanishing four-form flux, along the lines of [4, 3]. The additional constraint arising from the equations of motion in this case simply implies that the field strength four-form must be null. It is of interest to explore further our method by constructing more general solutions than the ones we have presented here including, in particular, cases where fluxes are turned on.

[4]For the comparison of (3.33) with (3.34), it is necessary to split the index $a = (i, \hat{\imath}, 7)$ and write the G_2-tensors according to the decomposition $\mathbf{7} \to \mathbf{3} + \mathbf{3} + \mathbf{1}$ [24] (In particular, see eq. (A.4)).

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A A general example in six dimensions

So far we have worked out explicit examples in various dimensions where the minimal choice (2.16) turned out to be valid. However, this is not always possible as we have seen in subsection 3.3. Let us consider now a six-dimensional class of metrics based on four-dimensional self-dual metrics with a translational Killing vector field $\partial/\partial\tau$. The general form of the metric is [13]

$$ds_4^2 = V(d\tau + \omega_i dx^i)^2 + V^{-1} dx_i^2 \ , \qquad i = 1, 2, 3 \ , \tag{A.1}$$

where the V and ω_i, $i = 1, 2, 3$ obey

$$\partial_i V^{-1} = -\epsilon_{ijk} \partial_j \omega_k \quad \Longrightarrow \quad \partial^2 V^{-1} = 0 \ . \tag{A.2}$$

Hence the most general multi-center solution for V^{-1} is

$$V^{-1} = V_0 + \sum_{a=1}^{N} \frac{m}{|\vec{x} - \vec{x}_a|} \ . \tag{A.3}$$

With the above choice, the singularities of the harmonic function V^{-1} at $\vec{x} = \vec{x}_a$ correspond to removable *nut* singularities of the metric (A.1) provided that the variable τ has period

$4\pi m$. This is the reason that the strength m for all of these singularities has be chosen to be the same. Hence, it follows that if the constant $V_0 \neq 0$ (in which case it can be normalized to 1), the space is asymptotically locally flat (ALF). If $V_0 = 0$ then the space is asymptotically locally Euclidean (ALE), with boundary at infinity S^3/R_N, where R_N is a discrete subgroup of $SO(4)$. The case of Taub–NUT we considered before corresponds to the one-center ALF space and that of the Eguchi–Hanson to the two-center ALE space. This solution preserves 1/2 of the maximal supersymmetry provided that the Killing spinor is constant and subject to the projection (3.4). Also, depending on the arrangement of the centers at $\vec{x} = \vec{x}_a$, part of the symmetry of the $\mathbb{R}^3$ space spanned by the x^i's is preserved by the metric (A.1).

According to (2.1), the ansatz for the six-dimensional metric with $SU(2) \ltimes \mathbb{R}^4$ holonomy is

$$d\hat{s}_6^2 = 2du\, dv + 2(V_i dx^i + V_\tau d\tau)du + F du^2 + V(d\tau + \omega_i dx^i)^2 + V^{-1} dx_i^2 \ , \qquad \text{(A.4)}$$

where F, V_τ and V_i are functions of u and of the x^i's. It is easy to see that in trying to introduce a u-dependence, the minimal choice (2.16) is not enough, since the condition (2.17) is not satisfied. Hence, we have to try and employ our method in full generality. We will use the frame basis

$$e^i = V^{-1/2} dx^i \ , \quad i = 1,2,3 \ , \qquad e^4 = V^{1/2}(d\tau + \omega_i dx^i) \ , \qquad \text{(A.5)}$$

and for convenience parameterize V_τ, V_i and F in terms of the functions Λ, H and Λ_i whose properties are easy to state

$$F = \Lambda + VH^2 \ , \qquad V_\tau = VH \ , \qquad V_i = \Lambda_i + VH\omega_i \ , \quad i = 1,2,3 \ . \qquad \text{(A.6)}$$

These auxiliary functions will be determined in order to satisfy the constraints (2.7), (2.8) and (2.15). We find that, Λ and Λ_i, like V^{-1}, are harmonic functions in $\mathbb{R}^3$, namely

$$\partial^2 \Lambda = \partial^2 \Lambda_i = 0 \qquad \text{(A.7)}$$

and in addition

$$\frac{dV^{-1}}{du} = \partial_i \Lambda_i \ . \qquad \text{(A.8)}$$

The function H is then determined from

$$\partial_i H = \epsilon_{ijk} \partial_j \Lambda_k + \dot{\omega}_i \ , \qquad \text{(A.9)}$$

whose integrability is guaranteed from (A.2), (A.7) and (A.8). Then computing the matrix Λ^{ab} from (2.7) we find

$$\Lambda^{i4} = H\partial_i V + V\epsilon_{ijk}\partial_j\Lambda_k \ , \qquad \Lambda^{ij} = V\partial_{[i}\Lambda_{j]} + H\epsilon_{ijk}\partial_k V \ . \tag{A.10}$$

The fact that $\Lambda^{4i} + \frac{1}{2}\epsilon_{ijk}\Lambda^{jk} = 0$, together with the projection (3.4), forces the condition (2.8) to be satisfied.

The harmonic functions Λ and Λ_i have forms similar to (A.3), with the important observation that the Λ_i's are linked to V^{-1}, as dictated by the condition (A.8). Taking this into account we summarize the result,

$$V^{-1} = V_0 + \sum_{a=1}^{N} \frac{m}{|\vec{x} - \vec{x}_a(u)|} \ , \qquad \Lambda_i = -\sum_{a=1}^{N} \frac{m\ \dot{x}_a^i(u)}{|\vec{x} - \vec{x}_a(u)|} \ . \tag{A.11}$$

Hence, we see that the fixed centers at $\vec{x} = \vec{x}_a$ can move with the (light-cone time) by becoming u-dependent. In addition

$$\Lambda = \sum_{a=1}^{N} \frac{\lambda_a(u)}{|\vec{x} - \vec{y}_a(u)|} \ , \tag{A.12}$$

where $\vec{x} = \vec{y}_a$ represent a set of, in principle, u-dependent centers, with the strength of each center denoted by the arbitrary functions $\lambda_a(u)$. We have also set the additive constants in Λ and Λ_i to zero since they can be always absorbed by shifts of the coordinate v in (2.1). Finally, the functions ω_i and H are computed using (A.2) and (A.9), respectively.

In order to investigate the properties of our metric (A.4), we have to specify the kind of motion of the centers of the harmonic functions. Let's consider the important class of such motions where the centers of V^{-1} assume definite fixed values at the remote past and the remote future light-cone time u, when also the strengths of the centers of Λ vanish. Namely,

$$\lim_{u\to\pm\infty} x_a(u) = x_a^{\pm} \ , \quad \lim_{u\to\pm\infty} \dot{x}_a(u) = 0 \ , \quad \lim_{u\to\pm\infty} \lambda_a(u) = 0 \ , \qquad a = 1, 2, \cdots, N \ . \tag{A.13}$$

Then, regularity of the solution near the centers in the remote past and feature requires that they are distinct and all have strength m. Therefore, our six-dimensional metric interpolates between two, generally different, multi-center metrics of the form (A.1) (times the two-dimensional light-cone). Since the arrangements of these centers could be different, the symmetries preserved by the solution can also differ as well.

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